B.Sc. Part III (Hons.) 5

METRIC SPACE (CONTD.)

Q. If $x = (x_1, x_2)$, $y = (y_1, y_2)$ belong to R^2, then prove that the function

$d : R^2 \times R^2 \to R$ defined as

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$ is metric on R^2.

Soln. We shall prove the conditions for a function to be metric.

i.e. (a) $d(x,y) \geq 0$

(b) $d(x,y) = 0 \Leftrightarrow x = y$

(c) $d(x,y) = d(y,x)$

(d) $d(x,y) \leq d(x,z) + d(z,y)$.

(a) $\because d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

Since, both $|x_1 - y_1|$ and $|x_2 - y_2|$ are non-negative.

$\Rightarrow$ their sum is also non-negative.

$\Rightarrow d(x,y) \geq 0$

(b) If $x = y \Leftrightarrow (x_1, x_2) = (y_1, y_2)$

$\Leftrightarrow x_1 = y_1$ and $x_2 = y_2$

$\Leftrightarrow x_1 - y_1 = 0$ and $x_2 - y_2 = 0$

$\Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0$

$\Leftrightarrow d(x,y) = 0$

Thus, $d(x,y) = 0 \Leftrightarrow x = y$.

(c) To prove $d(x,y) = d(y,x)$

$\because |x_1 - y_1| = |y_1 - x_1|$ and $|x_2 - y_2| = |y_2 - x_2|$

$\Rightarrow |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2|$

$\Rightarrow d(x,y) = d(y,x)$ ✓

(d) Let $z = (z_1, z_2)$

Now, let $\alpha_1 = |x_1 - z_1|$, $\alpha_2 = |x_2 - z_2|$

$\beta_1 = |z_1 - y_1|$, $\beta_2 = |z_2 - y_2|$

$\gamma_1 = |x_1 - y_1|$, $\gamma_2 = |x_2 - y_2|$

$\therefore \gamma_1 + \gamma_2 = |x_1 - y_1| + |x_2 - y_2|$

$= |(x_1 - z_1) + (z_1 - y_1)| + |(x_2 - z_2) + (z_2 - y_2)|$

$\leq |\alpha_1 + \beta_1| + |\alpha_2 + \beta_2| \leq |\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2|$

$\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|$

$d(x,y) \leq d(x,z) + d(z,y)$

Hence, all the conditions for a metric are satisfied. Thus, d is a metric.